Three-object two-bidder simultaneous auctions: chopsticks and tetrahedra

Balázs Szentes [*] and Robert W. Rosenthal [1]

Department of Economics, University of Chicago, 1126 East 59th Street, Chicago, IL 60637, USA

Received 31 October 2001

Abstract

Symmetric equilibria are constructed for a class of symmetric auction games. The games all have two identical bidders bidding in three simultaneous first-price sealed-bid auctions for identical objects. Information is complete and the bidders' marginal valuations increase for the second object and then decrease for the third. In all cases the support of the mixture that generates the equilibrium is two-dimensional, and it surrounds a three-dimensional set of best responses. This appears to be a previously unknown structure.
© 2003 Elsevier Science (USA). All rights reserved.

1. Introduction

Offshore oil leases and spectrum licenses are examples of objects sold by the US government through simultaneous auctions of one kind or another. Other examples of simultaneous auctions are abundant around the world. Situations for which a simultaneous design is often recommended are when a bidder's valuation for one object is typically dependent on what other objects the bidder may win. A simultaneous auction allows bidders to express their preferences over sets of objects through their bids, although it does not necessarily result in an assignment that is either efficient or revenue maximizing for the seller.

---

[*] Corresponding author.
E-mail address: szentes@uchicago.edu (B. Szentes).

[1] Bob Rosenthal passed away on the 27th of February 2002.

Though details of simultaneous design rules differ, they all tend to confront bidders with an "exposure problem": For example, a bidder whose valuation for a set of objects exceeds the sum of his valuations for the separate items that make up the set may be tempted to bid above the separate stand-alone valuations of the individual items in hopes of winning the set, risking having overbid on pieces in the event that the grand plan is unsuccessful. Various measures have been used to soften exposure problems: in spectrum auctions the simultaneous designs typically involve ascending prices, which allow bidders time to assess gradually the likelihood of successfully acquiring various combinations of spectrum blocks; and provisions for bid withdrawals are often included. These measures do not completely eliminate the problem, however, and the US FCC has recently announced that it will begin to allow all-or-none bids on subsets of licenses to try to solve the problem directly.[2]

Despite all the attempts to get around them, surprisingly little is known about the structure of equilibria in games generated by simultaneous auctions that present exposure problems.[3] This paper contributes to what is known in a small way: We construct equilibria for a small class of such auction games. These auctions all involve only three identical objects, two identical bidders, and complete information. But the class of auctions is a particularly important one nonetheless, because the bidders' marginal valuations are assumed to be first increasing and then decreasing in the number of objects acquired. For spectrum license auctions, this is a particularly relevant specification: in some such auctions (e.g., Netherlands 1998, Australia 2000) the spectrum blocks for sale were (by design) individually too small to be useful by themselves and the total available was much more than a single bidder could use efficiently (or was allowed to win).

As will be seen, not only was equilibrium structure of the games we analyze here previously unknown, but the structures turn out to be of a completely new form; we know of nothing similar in the literature. Furthermore we exhibit a new technique for equilibrium construction which makes use of a connection between familiar auction models that are difficult to analyze and unfamiliar ones that are easier. We believe that this technique may have a broader range of applicability than its use here.

The games in this paper are simultaneous "chopstick" auctions: three identical chopsticks are sold simultaneously, either in separate first-price sealed-bid auctions or in separate second-price sealed-bid auctions. There are two bidders, and it is common knowledge between them that a pair of chopsticks is worth \$2 but that a single chopstick is worth nothing by itself. A third chopstick is worth an additional \$$\alpha$, bringing the total value of a three-object set to \$$(2 + \alpha)$.[4] (As usual, we assume it is common knowledge between the bidders that both of them have risk-neutral preferences over money lotteries.) We call the cases where $\alpha = 0$ the "pure chopstick" cases;[5] the equilibria we find for

---

[2] See (Milgrom, 2000) for a discussion of some of the issues that the FCC is confronting.

[3] See, however, (Krishna and Rosenthal, 1996; Rosenthal and Wang, 1996).

[4] The parameter α can be negative. This need not be taken literally; it represents the case where the marginal value of the third object is less than that of the first, which has been normalized at \$0 for convenience. (If the marginal value of the first object is not zero, the equilibrium structure is modified in a simple way.)

[5] Credit goes to Mary Lucking-Reiley for the evocative chopstick name for identical objects that are useless except in pairs.

the pure-chopstick first- and second-price auctions are symmetric both across people and across chopsticks, have particularly simple and beautiful forms, and are valid for all tie-break rules. When $\alpha \neq 0$, the validity of the constructions depends on specific tie-break rules that are tied to the specific α values. We suspect that no equilibria exist when the tie-break rules are other than the ones that we need to specify, and we suspect that there are no equilibria other than the ones we have found even under the "right" rules.[6]

While the sealed-bid nature of our auction games makes them atypical of real spectrum auctions, there are often backup rules in real spectrum auctions that give the auctioneer the right to call for a final sealed-bid round in certain circumstances (and therefore for which plans must be made by bidders). Simplicity is, however, the main reason for studying sealed-bid auctions before attempting to understand ascending ones. Another unrealistic feature of the chopstick examples is the assumption that the bidders' valuation schedules are the same and that this is common knowledge. Again, we opt for simplicity as a research strategy.

Pure strategies in our games are triples of real numbers (bids) and mixed strategies are therefore probability distributions over R^3. In the pure chopstick cases ($\alpha = 0$), the supports of the mixtures that generate the symmetric equilibria in both the first- and second-price cases, turn out to be the surfaces of regular tetrahedra, and the distributions themselves turn out to be uniform on these surfaces. In addition, in each case all the points inside the tetrahedron are pure best responses to the equilibrium mixture. The combination of the support being the surface of a polyhedron and the set of best responses being the entire polyhedron is completely unfamiliar to us.

When $\alpha > 1$ and the tie-break rule in case of a tie for all three objects simultaneously at bids of $(2+\alpha)/3$ awards all objects to the same bidder, in both the first- and second-price case there is obviously a pure-strategy symmetric equilibrium in which both bidders bid $(2+\alpha)/3$ on each object.

When $-\infty < \alpha < 1$ but $\alpha \neq 0$, the constructions become more complicated because the tie-break rule and the equilibrium mixture need to be constructed jointly and ties occur in equilibrium with probability strictly between zero and one. To accomplish the task we introduce a new, indirect construction method. First we establish a functional-form relationship between the symmetric equilibria together with tie-break rules of our chopstick auctions on the one hand and the symmetric equilibria of another class of simultaneous sealed-bid auction games for which the tie-break rules turn out not to be critical on the other hand. Then we construct equilibria for these other games. For these games the equilibrium mixtures are again supported on the surfaces of symmetric tetrahedra; but the tetrahedra are no longer regular ones, and the distributions, though symmetric, are no longer uniform, although they are uniform when restricted to single faces of the tetrahedra. The tetrahedra and distributions then become deformed when mapped back into equilibria

---

[6] There are two approaches to existence theorems for games like ours. One approach (Dasgupta and Maskin, 1986; Reny, 1999, for example) provides sufficient conditions for existence that depend on the specifics of the tie-break rules and that are sometimes difficult to check in practice. The other approach is that of (Simon and Zame, 1990), where existence is proved for at least one (unspecified) tie-break rule. Jackson, et al. (2001) extends the Simon–Zame approach to incomplete-information models in which players can provide information beyond bids that is used to break ties.

for the original games of interest. The entire equilibrium structure changes continuously with α, approaching the pure-chopstick equilibria in both cases as α approaches zero.

We begin in Section 2 with the construction of the equilibria for both first- and second-price auctions in the pure chopstick case. Section 3 is devoted to the indirect construction for both cases when $0 < \alpha < 1$. Section 4 treats both cases when $\alpha < 0$. Section 5 concludes with a discussion of equilibria of related games. Some proofs are relegated to an Appendix A.

2. The pure chopstick case

A pure strategy in any of the chopstick-auction games is a triple of real numbers whose ith coordinate is interpreted as a bid for the ith object. A mixed strategy is therefore a probability measure over R^3. Let the tetrahedron T be defined as the convex hull of the four points: $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(0, 0, 0)$. Alternatively,

$$T = \big\{(x, y, z)\colon x + y + z \leqslant 2,\ x \leqslant y + z,\ y \leqslant x + z,\ z \leqslant x + y\big\}.$$

Theorem 1. *The uniform probability measure on the surface S of T generates a symmetric equilibrium of the pure chopstick first-price auction game (i.e., when $\alpha = 0$).*

Most of the rest of this section is devoted to a proof of this theorem—showing that each point in S is a best response to the uniform probability measure on S. In fact, we show also that each point in T is a best response. At the end of the section we construct the equilibrium for the second-price pure-chopstick game as an easy corollary.

Proof of Theorem 1. First some notation and easy preliminary observations. The cumulative distribution function of the probability measure in question is denoted G. Obviously, $G(1, 1, 1) = 1$ and $G(0, 0, 0) = 0$. The four faces that comprise S are congruent equilateral triangles. Three of them (those that touch $(0, 0, 0)$) possess the property that one of the bids is the sum of the other two; we denote these faces $F1$, $F2$, and $F3$, respectively, where the integer designates the coordinate of the bid that is the sum of the other two bids. The fourth face Ft is the "top" of T; its triples all have the property that the sum of the three bids is two.

Figure 1 depicts the projections of the Fi onto the (x, y)-subspace. Note that the hypotenuse of each of the (right-triangle) projections forms one of the main diagonals of the square. The right angle of the projection of $F1$ is located at $(1, 0)$, that of the projection of $F2$ is at $(0, 1)$, that of the projection of $F3$ is at $(0, 0)$, and that of the projection of Ft is at $(1, 1)$. Note also that the area of any polygon in Fi is proportional to the area of its projection and that the constant of proportionality is the same across i. This is because the four faces all happen to lie in planes that intersect the coordinate plane $\{(x, y, 0)\colon (x, y) \in R^2\}$ at the same angle.

We will show that against the distribution in question the expected profit of any bid triple in T is the constant 0. First note that the expected profit of the bid triple (a, b, c) against the distribution function G is
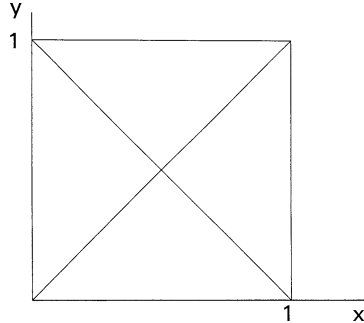
Fig. 1. Projections of Fi, $i = 1, 2, 3, t$.

$$\pi(a,b,c) \equiv 2G(a,b,1) + 2G(a,1,c) + 2G(1,b,c) - 4G(a,b,c) - aG(a,1,1)$$
$$- bG(1,b,1) - cG(1,1,c). \tag{1}$$

The first three terms in the definition of π are winning probabilities for specific bid pairs, each multiplied by two, the value of winning two objects. The fourth term corrects for the multiple countings in the first three terms of the probability that all three bids win. The last three terms give the expected payments for winning bids.

Next, under G, although the three bids are not mutually independent, it turns out that each pair of them is independently distributed and that the marginal distributions on single bids are all uniform on $[0, 1]$. To see this, note that for the pair $(a, b) \in [0, 1]^2$ the probability that both a and b are winning bids is proportional to the sum of the areas of the intersections of the brick $[0, a] \times [0, b] \times [0, 1]$ with the four faces of T. But these are in turn proportional to the sums of the areas of the intersections of $[0, a] \times [0, b]$ with the four triangles in Fig. 1. Now these four triangles may be paired into mutually exclusive regions that cover $[0, 1]^2$ exactly, so that the probability of the brick $[0, a] \times [0, b] \times [0, 1]$ is proportional to the area ab of its projection onto the space of its first two coordinates. Hence it must be that $G(a, b, 1) = ab$ on $[0, 1]^2$ and $G(a, 1, 1) = a$ on $[0, 1]$. So by symmetry we have

$$\pi(a,b,c) = 2ab + 2bc + 2ac - a^2 - b^2 - c^2 - 4G(a,b,c). \tag{2}$$

The calculation of $G(a, b, c)$ requires a little more work; in particular its algebraic expression differs according to whether (a, b, c) is an element of T or not. As before, we shall construct $G(a, b, c)$ by summing the probabilities of the intersections of $[0, a] \times [0, b] \times [0, c]$ with the four faces of T. We begin by considering the case of $(a, b, c) \in T$.

First note that if $(a, b, c) \in T$, the intersection of $[0, a] \times [0, b] \times [0, c]$ with Ft is either the empty set or a singleton, so its probability is zero. Let PFi $(i = 1, 2, 3)$ denote the projection of Fi onto the subspace of its two smaller coordinates. We want to calculate the areas of the PFi-projections of the intersections of the brick with each of the three faces, respectively, because the projections are all proportional to the actual areas and the constant of proportionality is again the same.

Now assume without loss of generality that $a \geqslant b \geqslant c$.

Case 1, $F3$. In this case, a and b win whenever c does ($a \geqslant b \geqslant c \geqslant x + y$ implies $\min\{a, b\} \geqslant \max\{x, y\}$); so we seek the area of the triangle whose projection is shaded in Fig. 2, and which is easily seen to be $c^2/2$. (Of course, either a or b may be matched against x.)

Case 2, $F2$. In this case a wins whenever b does ($a \geqslant b \geqslant x + y$ implies $a \geqslant \max\{x, y\}$); and we seek the area of the trapezoid shaded in Fig. 3, and which is $(b^2 - (b - c)^2)/2$. (Either a or c can be matched against x.)

Case 3, $F1$. In this case, we seek the area of the pentagon shaded in Fig. 4, and which is $bc - (c - (a - b))^2/2$. (Either b or c can be matched against y.) It is important to notice that for (a, b, c) to be an element of T, b and c must be such that the rectangle $[0, b] \times [0, c]$ is cut by the line $\{(y, z): y + z = a\}$.

Adding the three projected areas produces

$$ab + bc + ac - \frac{a^2 + b^2 + c^2}{2}.$$

But each of the four PFi has total area $1/2$, so the expression above must be halved to produce $G(a, b, c)$, and, plugging this into (2), we conclude that $\pi(a, b, c) = 0$ on T.
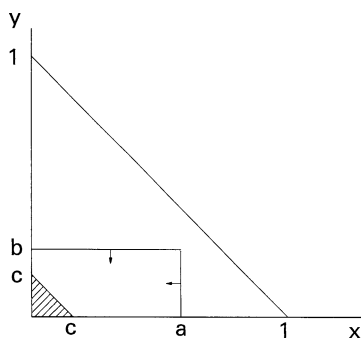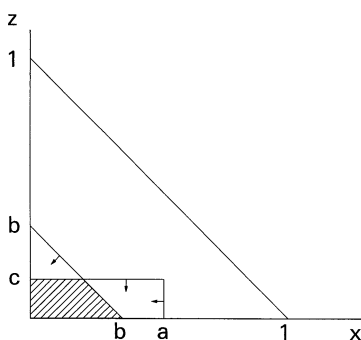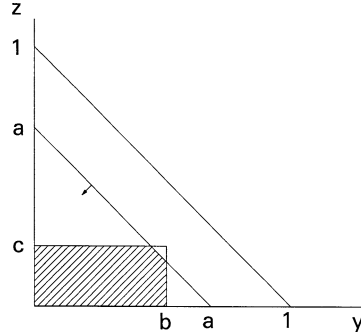

Fig. 2. Case 1: $F3$.

Fig. 3. Case 2: $F2$.

Fig. 4. Case 3: $F1$.

Outside T, weak dominance considerations allow us to restrict consideration to the unit cube. If $(a, b, c) \in [0, 1]^3 \backslash T$, there are two cases to consider:

Case a: $a + b + c \leqslant 2$ but one of the bids exceeds the sum of the other two; and
Case b: $a + b + c > 2$ but none of the bids exceeds the sum of the other two.

For both cases, the polynomial expression for π is the same as that given in (2); the only difference is in the calculation of $G(a, b, c)$.

Case a. Again the intersection of $[0, a] \times [0, b] \times [0, c]$ with Ft has zero probability. By the convention that $a \geqslant b \geqslant c$, it is the a bid that exceeds the sum of the other two, and this alters the $G(a, b, c)$ only in Case 3, where it is increased by the outward shifting of the diagonal boundary of the pentagon. But this increased value of $G(a, b, c)$ only drives π below zero and so such bid triples are not profitable.

Case b. Now the intersection of $[0, a] \times [0, b] \times [0, c]$ with Ft has positive probability, but the other three intersections are the same as in Cases 1, 2, and 3, so $G(a, b, c)$ is again increased and π is again negative.   □

The construction for the second-price pure chopstick case is very similar. Let $2T$ denote the convex hull of the four points $(2, 2, 0)$, $(2, 0, 2)$, $(0, 2, 2)$, and $(0, 0, 0)$.

Corollary 2. *The uniform probability measure on the surface of* $2T$ *generates a symmetric equilibrium of the pure chopstick second-price auction game.*

Proof. Let G' be the cdf of this uniform measure. The expected profit of the pure triple $(a, b, c) \in 2T$ is now

$$\pi'(a, b, c) \equiv 2G'(a, b, 1) + 2G'(a, 1, c) + 2G'(1, b, c) - 4G'(a, b, c)$$
$$- \frac{a}{2}G'(a, 1, 1) - \frac{b}{2}G'(1, b, 1) - \frac{c}{2}G'(1, 1, c), \tag{3}$$

since the expected payment of a winning bid against a uniformly distributed losing bid is half the winning bid. But $G'(a, b, c) = G(a/2, b/2, c/2)$ and so $\pi'(a, b, c) =$

$\pi(a/2, b/2, c/2) = 0$. The argument that there are no profitable triples in $[0, 2]^3 \setminus 2T$ is similar to the analogous part of the proof of Theorem 1. □

We will continue to exploit the close relationships between the equilibria of first- and second-price auctions in subsequent sections. In addition, the relationships between both of these and auctions with artificial payment rules will be the key to all the "impure" chopstick auction development.

3. The case $0 < \alpha < 1$

Let a "payment rule" $p : R_+ \to R_+$ be a nondecreasing, continuous function (of a single bid) with the property that $\lim_{x \to \infty} p(x) = \infty$, and define the generalized inverse function $p^{-1}(x) = \sup\{x' : p(x') = x\}$. Consider artificial variants of the first-price chopstick auctions which differ only in that the winning bidder pays $p(x)$ instead of x when a bid of x wins any of the objects. Denote by $\Gamma(\alpha, p)$ the game defined by marginal third value α and payment rule p. Of course, if I is the identity function, then $\Gamma(\alpha, I)$ is the original first-price chopstick auction defined by the parameter α. Tie-break rules are left unspecified for now; we are not necessarily assuming the same tie-break rule in both $\Gamma(\alpha, p)$ and $\Gamma(\alpha, I)$. More notation: $\mathbf{p}(x, y, z) \equiv (p(x), p(y), p(z))$. Similarly for $\mathbf{p}^{-1}$.

Proposition 3. *If G_p is a cumulative distribution function having atomless marginal distributions that generates a symmetric equilibrium for $\Gamma(\alpha, p)$ with some tie-break rule, then $G_I \equiv G_p \circ \mathbf{p}^{-1}$ generates a symmetric equilibrium for $\Gamma(\alpha, I)$ with some tie-break rule.*

Since ties occur with probability zero against G_p, the expected profit of the bid triple (x, y, z) against G_p in $\Gamma(\alpha, p)$ is

$$\begin{aligned}
\pi_p(x, y, z) &\equiv 2G_p(x, y, \infty) + 2G_p(x, \infty, z) + 2G_p(\infty, y, z) \\
&\quad - (4 - \alpha)G_p(x, y, z) - p(x)G_p(x, \infty, \infty) - p(y)G_p(\infty, y, \infty) \\
&\quad - p(z)G_p(\infty, \infty, z).
\end{aligned}$$

When p is strictly monotone, the expected profit of the bid triple $\mathbf{p}(x, y, z)$ against G_I in $\Gamma(\alpha, I)$ is evidently

$$\begin{aligned}
\pi_I\big(\mathbf{p}(x, y, z)\big) &= 2G_p(x, y, \infty) + 2G_p(x, \infty, z) + 2G_p(\infty, y, z) \\
&\quad - (4 - \alpha)G_p(x, y, z) - p(x)G_p(x, \infty, \infty) - p(y)G_p(\infty, y, \infty) \\
&\quad - p(z)G_p(\infty, \infty, z),
\end{aligned}$$

so any image under $\mathbf{p}$ of a maximizer of π_p is a maximizer of π_I and the argument is complete. The complications come with flats in p, and this is where the tie-break rule for $\Gamma(\alpha, I)$ becomes relevant. For a complete proof, which identifies the class of usable tie-break rules, see Appendix A.

For each $\alpha \in (0, 1)$ our goal is to come up with a choice of payment rule p for which we can compute a symmetric equilibrium of $\Gamma(\alpha, p)$. Toward this end, let $\beta \equiv \alpha/(2 - \alpha)$ and define a tetrahedron analogously to T in Section 2 by

$$T_\beta \equiv \left\{ (x, y, z): x + y + z \leqslant 2 + \beta, \ x \leqslant \frac{y + z}{1 + \beta}, \ y \leqslant \frac{x + z}{1 + \beta}, \ z \leqslant \frac{x + y}{1 + \beta} \right\}.$$

(Alternatively, T_β is the convex hull of the four points $(1, 1, \beta)$, $(1, \beta, 1)$, $(\beta, 1, 1)$, and $(0, 0, 0)$. Intuitively, the closer is α to one: the higher is β and the slimmer is T_β.) Each of the four faces of the surface S_β of T_β is identified as Fi, $i = 1, 2, 3, t$, using the code that is the obvious analogue to that in Section 2, and for $i \neq t$ the projections onto the space of the two smaller coordinates are again identified, respectively, as PFi. Now let μ_β be the unique probability measure on S_β:

(i)   which is uniform on all four faces separately;
(ii) which assigns equal total measure to the faces $F1$, $F2$, and $F3$; and
(iii) whose restriction to $F3 \cup Ft$ projected onto the $PF3$-plane is uniform.[7] (This projection is depicted in Fig. 5.)

Proposition 4. *For $\alpha \in (0, 1)$, the probability measure μ_β generates a symmetric equilibrium for $\Gamma(\alpha, p_\beta)$ for all tie-break rules, where*

$$p_\beta(x) = \begin{cases} \dfrac{2 - \beta - \beta^2}{\frac{1}{\beta} - \beta + \frac{1}{2} - \frac{\beta^2}{2}} & \text{if } 0 \leqslant x < \beta, \\[2em] \dfrac{x^2(2 - \beta - \beta^2)}{x(2 + \beta) - x^2\left(\beta + \frac{\beta^2}{2}\right) - \beta - \frac{\beta^2}{2}} & \text{if } \beta \leqslant x < 1, \\[1.5em] x & \text{if } 1 \leqslant x. \end{cases}$$

Notice first that this payment rule satisfies the assumptions of Proposition 3 and second that it is constant on $[0, \beta]$. Since μ_β has no atoms, ties occur with probability zero at
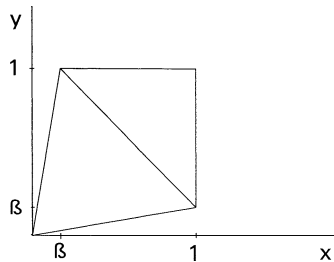

Fig. 5. Projection of $F3 \cup Ft$.

---

[7] Condition (iii) corrects for the fact that when $\alpha \neq 0$, Ft no longer possesses the same symmetric relationship to the other Fi as when $\alpha = 0$.
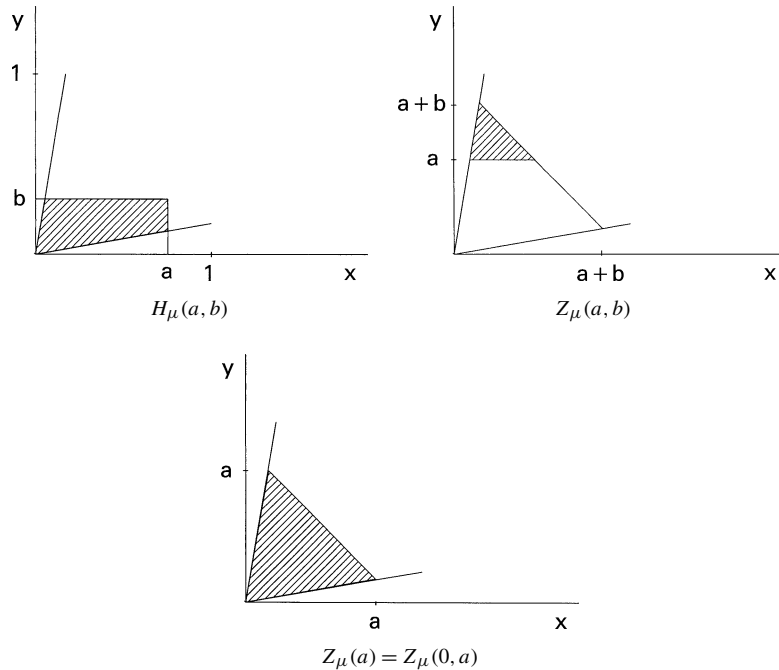
$H_\mu(a, b)$

$Z_\mu(a, b)$

$Z_\mu(a) = Z_\mu(0, a)$

Fig. 6.

the equilibrium of $\Gamma(\alpha, p_\beta)$. But since the marginals of μ_β assign positive probability to $[0, \beta)$, there will be ties with positive probability at the induced equilibrium of $\Gamma(\alpha, I)$.

Before proving Proposition 4, we need some additional notation. If μ is any measure on S_β, let $P\mu$ denote the projection onto the $PF3$-plane of its restriction to $F3 \cup Ft$. Now for $(a, b) \in R_+^2$, let $H_\mu(a, b)$ be the $P\mu$-measure of $([0, a] \times [0, b])$; let $Z_\mu(a, b)$ be the $P\mu$-measure of the triangle generated by the points $(\beta a, a)$, $(\beta(a + b), a + b)$, and $(\beta a + (1 + \beta)b, a)$; and let $Z_\mu(a) = Z_\mu(0, a)$ be the $P\mu$-measure of the triangle generated by the points $(0, 0)$, $(a, \beta a)$, and $(\beta a, a)$. (See Fig. 6, the panels of which depict the polygons whose measures are specified above.[8])

The heart of the proof consists of a series of lemmas. Lemma 5 states that the sum of the first four terms of the profit function relative to *any* symmetric probability measure μ on S_β can be expressed as the sum of terms that are each functions only of pairs of the three bids.

Lemma 5. *If J_μ is the distribution function associated with any symmetric probability measure μ supported on S_β and*

$$Q(a, b) \equiv 2H_\mu(a, b) + \frac{\alpha}{2}\big(Z_\mu(a) + Z_\mu(b)\big) + (2 - \alpha)Z_\mu(b, a - b),$$

then for any $(a, b, c) \in T_\beta$

---

[8] The depictions in Fig. 6 assume $1 > a > b$ and $a > \beta(a + b)$.

$$Q(a, b) + Q(a, c) + Q(b, c)$$
$$= 2J_\mu(a, b, \infty) + 2J_\mu(a, \infty, c) + 2J_\mu(\infty, b, c) - (4 - \alpha)J_\mu(a, b, c).$$

Proof. See Appendix A. □

Lemma 6 states that in the special case of the distribution function G_{μ_β} generated by μ_β, the same profit terms can be separated into functions of the individual variables.[9]

Lemma 6. *If $V(a) = 2H_{\mu_\beta}(a, a) + \alpha Z_{\mu_\beta}(a)$ and $(a, b, c) \in T_\beta$, then*

$$V(a) + V(b) + V(c) = 2G_{\mu_\beta}(a, b, \infty) + 2G_{\mu_\beta}(a, \infty, c) + 2G_{\mu_\beta}(\infty, b, c)$$
$$- (4 - \alpha)G_{\mu_\beta}(a, b, c).$$

Proof. See Appendix A. □

The final lemma lists the functional forms needed in the proof of the proposition.

Lemma 7. $Z_{\mu_\beta}(x) = \dfrac{x^2(1 - \beta^2)}{2(2 - \beta - \beta^2)}, \qquad H_{\mu_\beta}(x, x) = \dfrac{x^2(1 - \beta)}{2 - \beta - \beta^2},$

$$G_{\mu_\beta}(x, \infty, \infty) = \begin{cases} \dfrac{x^2\left(\frac{1}{\beta} - \beta + \frac{1}{2} - \frac{\beta^2}{2}\right)}{2 - \beta - \beta^2} & \text{if } 0 \leqslant x < \beta, \\[3mm] \dfrac{x(2 + \beta) - x^2\left(\beta + \frac{\beta^2}{2}\right) - \beta - \frac{\beta^2}{2}}{2 - \beta - \beta^2} & \text{if } \beta \leqslant x < 1, \\[3mm] 1 & \text{if } 1 \leqslant x. \end{cases}$$

Proof. The proof of Lemma 7 follows directly from the definitions. □

Finally we come to

Proof of Proposition 4. First, if $(x, y, z) \in T_\beta$, then from Lemma 6

$$\pi_{p_\beta}(x, y, z) = V(x) + V(y) + V(z) - p_\beta(x)G_{\mu_\beta}(x, \infty, \infty)$$
$$- p(y)G_{\mu_\beta}(\infty, y, \infty) - p(z)G_{\mu_\beta}(\infty, \infty, z).$$

But for $0 \leqslant x < 1$, $V(x) = p_\beta(x)G_{\mu_\beta}(x, \infty, \infty) = x^2$ and so π_{p_β} is identically zero on the tetrahedron.

For $(x, y, z) \in [0, 1]^3 \backslash T_\beta$, there are, as in Section 2, two cases.

Case a: $x + y + z \leqslant 2 + \beta$, but one of the bids exceeds the sum of the other two multiplied by $(1 + \beta)^{-1}$; and

---

[9] We have decomposed the separability argument into two steps (Lemmas 5 and 6) to highlight that the initial step goes through in more general settings. This observation may prove useful beyond the present setting.

Case b: $x + y + z > 2 + \beta$ but none of the bids exceeds the sum of the other two multiplied by $(1 + \beta)^{-1}$.

Case a. Assume that it is the condition $x \leqslant (y + z)/(1 + \beta)$ that is violated. There are two possibilities: First if y or z is smaller than β and $x \geqslant \beta^{-1}y$, then $H_\mu(x, y)$ must be replaced in Lemma 5 by $H_\mu(\beta^{-1}y, y)$, which is strictly smaller and results in smaller values for $\pi_{p_\beta}(x, y, z)$. Otherwise, the calculation of $J_\mu(x, y, z)$ in the proof of Lemma 5 must be increased as in Case a of Theorem 1 for the pure chopstick case. As there, the result is a decrease in profits.

Case b. Assume now that the condition $x + y + z \leqslant 2 + \beta$ is violated. In this case the bid triple sometimes wins all three objects against Ft. As in Case b of the proof of Theorem 1, this reduces profits.   □

Propositions 3 and 4 immediately lead to

Theorem 8. *For $0 < \alpha < 1$, if p_β is defined as in Proposition 4, then $G_{\mu_\beta} \circ \mathbf{p}_\beta^{-1}$ generates a symmetric equilibrium of the first-price auction $\Gamma(\alpha, I)$ for some class of tie-break rules.*

Once again, the second-price auction construction can be found by modifying the first-price construction.

Theorem 9. *Let $q_\beta(x) = \partial V(x)/\partial G_{\mu_\beta}(x, \infty, \infty)$. Then $G_{\mu_\beta} \circ \mathbf{q}_\beta^{-1}$ generates a symmetric equilibrium of the second-price auction for some class of tie-break rules.*

Proof. First observe that

$$
q_\beta(x) = \begin{cases} \dfrac{2 - \beta - \beta^2}{\frac{1}{\beta} - \beta + \frac{1}{2} - \frac{\beta^2}{2}} & \text{if } 0 \leqslant x < \beta, \\[2ex] \dfrac{2x(2 - \beta - \beta^2)}{2 + \beta - 2x\left(\beta + \frac{\beta^2}{2}\right)} & \text{if } \beta \leqslant x < 1; \end{cases}
$$

so q_β, like p_β, is continuous, constant on $[0, \beta)$, and increasing on $[\beta, 1]$. Hence we can define the tie-break rule in the second-price auction between the bid vectors $\mathbf{q}_\beta(\mathbf{x})$ and $\mathbf{q}_\beta(\mathbf{y})$ to be the same as between $\mathbf{p}_\beta(\mathbf{x})$ and $\mathbf{p}_\beta(\mathbf{y})$ in the first-price auction. Next, in Proposition 4 we have shown that the probability measure μ_β generates a symmetric equilibrium in $\Gamma(\alpha, p_\beta)$. Here the players are again randomizing according to the same distribution, but instead of bidding (x, y, z), or $\mathbf{p}_\beta(x, y, z)$ as in $\Gamma(\alpha, I)$, they bid $\mathbf{q}_\beta(x, y, z)$. If they do so, the revenue parts of their payoffs are clearly the same as in $\Gamma(\alpha, p_\beta)$, so we only have to show that the cost parts are also the same. In $\Gamma(\alpha, p_\beta)$ the expected cost of the x bid was

$$
p_\beta(x)G_{\mu_\beta}(x, \infty, \infty) = V(x).
$$

The expected cost of bidding $q_\beta(x)$ in the second-price auction against $\mu_\beta \circ \mathbf{q}_\beta^{-1}$ is clearly

$$\int_0^x q_\beta(y)\, dG_{\mu_\beta}(y,\infty,\infty) = \int_0^x \frac{\partial V(y)}{\partial G_{\mu_\beta}(y,\infty,\infty)}\, dG_{\mu_\beta}(y,\infty,\infty).$$

But this is just $V(x) - V(0) = V(x)$ by the Fundamental Theorem of Calculus.

That no deviations are profitable is similar to the argument in the proof of Proposition 3. □

4. The case $\alpha < 0$

Since $\alpha < 0$, there is a preliminary issue of whether negative bids should be allowed. As it turns out, even when they are allowed they are not used in the equilibrium we construct; so it does not matter, but for formality's sake, we allow them. Next notice that the sign of α was immaterial to the proof of Proposition 3, which will play the same role here as in Section 3.

Now, for each $\alpha \in (-\infty, 0)$ define β, T_β, and μ_β exactly as in Section 3. We retain the same notation for the faces and projections of faces of the tetrahedron. Note that β is always between -1 and 0, so three of the extreme points of T_β are now outside R_+^3. The projection onto the $PF3$-plane of the restriction of μ_β to $F3 \cup Ft$ is depicted in Fig. 7. Intuitively, the closer is α to $-\infty$: the closer is β to -1 and the thicker is T_β.

Proposition 10. *For $\alpha \in (-\infty, 0)$, the probability measure μ_β generates a symmetric equilibrium for $\Gamma(\alpha, \widehat{p}_\beta)$ for all tie-break rules, where*

$$\widehat{p}_\beta(x) = \begin{cases} 0 & \text{if } -\infty \leqslant x < 0, \\ \dfrac{x^2(2 - \beta - \beta^2)}{x(2+\beta) - x^2\left(\beta + \frac{\beta^2}{2}\right) - \beta - \frac{\beta^2}{2}} & \text{if } 0 \leqslant x < 1, \\ x & \text{if } 1 \leqslant x. \end{cases}$$

Notice first that this payment rule satisfies the assumptions of Proposition 3 and second that it is constant on $[-\infty, 0)$. Notice also its similarity to the payment rule p_β in Section 3.

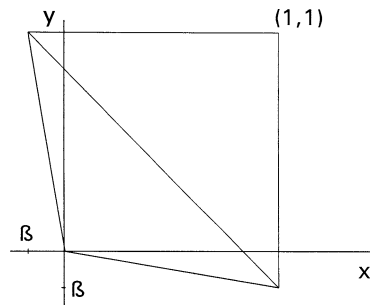

Fig. 7. Projection of $F3 \cup Ft$.

The proof of Proposition 10 follows that of Proposition 4 with a couple of exceptions. Now, for $(a, b) \in R_+^2$, $H_\mu(a, b)$ is the $P\mu$-measure of $([\beta, a] \times [\beta, b])$, $Z_\mu(a, b)$ is the $P\mu$-measure of the triangle generated by the points $(\beta a, a)$, $(\beta(a + b), a + b)$, and $(\beta a + (1 + \beta)b, a)$, and $Z_\mu(a)$ is the $P\mu$-measure of the triangle generated by the points $(0, 0)$, $(a, \beta a)$ and $(\beta a, a)$. (See Fig. 8, which includes depictions for some cases in which one of the arguments is negative.)

The statements and proofs of Lemmas 11 and 12 are now identical to those of Lemmas 5 and 6, respectively. (We provide copy of the statements, but not the proofs, below for

$H_\mu(a, b)$ $(a > b > 0)$

$H_\mu(a, b)$ $(a > 0 > b)$

$Z_\mu(a, b)$ $(a > 0)$

$Z_\mu(a, b)$ $(a < 0)$
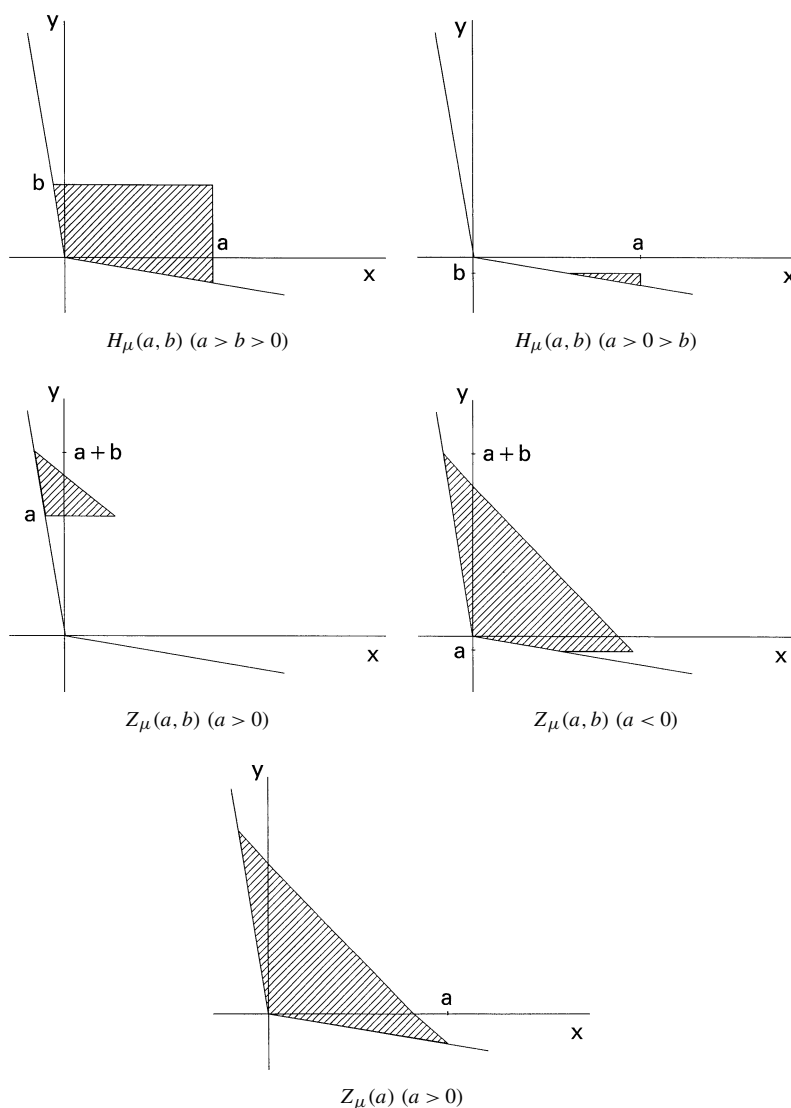
$Z_\mu(a)$ $(a > 0)$

Fig. 8.

convenience. But the reader should be careful to note that the geometry behind the algebra has different forms when one of the bids is negative.)

Lemma 11. *If J_μ is the distribution function associated with any symmetric probability measure μ supported on T_β and*

$$Q(a, b) \equiv 2H_\mu(a, b) + \frac{\alpha}{2}\big(Z_\mu(a) + Z_\mu(b)\big) + (2 - \alpha)Z_\mu(b, a - b),$$

then for any $(a, b, c) \in T_\beta$

$$\begin{aligned} Q(a, b) + Q(a, c) + Q(b, c) \;=\; & 2J_\mu(a, b, \infty) + 2J_\mu(a, \infty, c) \\ & + 2J_\mu(\infty, b, c) - (4 - \alpha)J_\mu(a, b, c). \end{aligned}$$

Lemma 12. *If $V(a) = 2H_{\mu_\beta}(a, a) + \alpha Z_{\mu_\beta}(a)$ then*

$$\begin{aligned} V(a) + V(b) + V(c) \;=\; & 2G_{\mu_\beta}(a, b, \infty) + 2G_{\mu_\beta}(a, \infty, c) \\ & + 2G_{\mu_\beta}(\infty, b, c) - (4 - \alpha)G_{\mu_\beta}(a, b, c). \end{aligned}$$

Lemma 13 parallels Lemma 7. The only difference is in the algebraic form of G_{μ_β}.

Lemma 13. $Z_{\mu_\beta}(x) = \dfrac{x^2(1 - \beta^2)}{2(2 - \beta - \beta^2)}, \qquad H_{\mu_\beta}(x, x) = \dfrac{x^2(1 - \beta)}{2 - \beta - \beta^2},$

$$G_{\mu_\beta}(x, \infty, \infty) = \begin{cases} \textit{irrelevant} & \textit{if } -\infty \leqslant x < 0, \\[2mm] \dfrac{x(2 + \beta) - x^2\big(\beta + \frac{\beta^2}{2}\big) - \beta - \frac{\beta^2}{2}}{2 - \beta - \beta^2} & \textit{if } 0 \leqslant x < 1, \\[2mm] 1 & \textit{if } 1 \leqslant x. \end{cases}$$

As with Lemma 7, the proof of Lemma 13 follows directly from the definitions. (The reason it is not necessary to know $G_{\mu_\beta}(x, \infty, \infty)$ when $x < 0$ is that $\widehat{p}_\beta(x) = 0$ there.)

Proof of Proposition 10. First, if $(x, y, z) \in T_\beta$, then from Lemma 12

$$\begin{aligned} \pi_{\widehat{p}_\beta}(x, y, z) \;=\; & V(x) + V(y) + V(z) - \widehat{p}_\beta(x)G_\beta(x, \infty, \infty) \\ & - \widehat{p}_\beta(y)G_\beta(\infty, y, \infty) - \widehat{p}_\beta(z)G_\beta(\infty, \infty, z). \end{aligned}$$

But for $0 \leqslant x < 1$, $V(x) = \widehat{p}_\beta(x)G_\beta(x, \infty, \infty) = x^2$ and so $\pi_{\widehat{p}_\beta}$ is identically zero on the tetrahedron.

For $(x, y, z) \in [0, 1]^3 \setminus T_\beta$ there are, as in Sections 2 and 3, two cases. The arguments in both are identical to those in the proof of Proposition 4.

Case a: $x + y + z \leqslant 2 + \beta$, but one of the bids exceeds the sum of the other two multiplied by $(1 + \beta)^{-1}$; and

Case b: $x + y + z > 2 + \beta$ but none of the bids exceeds the sum of the other two multiplied by $(1 + \beta)^{-1}$.

Case a. Assume that it is the condition $x \leqslant (y + z)/(1 + \beta)$ that is violated. There are two possibilities: First if y or z is smaller than β and $x \geqslant \beta^{-1}y$, then $H_\mu(x, y)$ must be replaced in Lemma 11 by $H_\mu(\beta^{-1}y, y)$, which is strictly smaller and results in smaller values for $\pi_{\widehat{p}_\beta}(x, y, z)$. Otherwise, the calculation of $J_\mu(x, y, z)$ in the proof of Lemma 11 must be increased as in Case a of Theorem 1 for the pure chopstick case. As there, the result is a decrease in profits.

Case b. Assume now that the condition $x + y + z \leqslant 2 + \beta$ is violated. In this case the bid triple sometimes wins all three objects against Ft. As in Case b of the proof of Theorem 1, this reduces profits. □

Propositions 3 and 10 immediately lead to

Theorem 14. *For $-\infty < \alpha < 0$, if $\widehat{p}_\beta$ is defined as in Proposition* 10*, then $G_\beta \circ \widehat{\mathbf{p}}_\beta{}^{-1}$ generates a symmetric equilibrium of the first-price auction for some class of tie-break rules.*

Again the second-price auction construction can be found by modifying the first-price construction. The proof parallels the proof of Theorem 9 and is omitted.

Theorem 15. *If $\widehat{q}(x) = \partial V(x)/\partial G_{\mu_\beta}(x, \infty, \infty)$ then $G_{\mu_\beta} \circ \widehat{\mathbf{q}}^{-1}$ generates a symmetric equilibrium of the second price auction for some class of tie-break rules.*

5. Discussion

We have constructed symmetric equilibria for a class of complete-information, simultaneous sealed-bid auction games with two bidders, three identical objects, and marginal valuations that increase at first and then decrease. The method is to find tetrahedron-based symmetric equilibria for a related class of artificial auctions and then to transform the mixtures in a particular way. The method works both when the game of interest is a first- or second-price auction.

There are two main novelties in this. One is the establishment of a useful relationship between different auctions through the "payment rule." By choosing just the right payment rules, simple equilibria can be found and transformed. This turns out to be a useful discovery for a much wider class of auction models, including incomplete-information auctions. Although in some ways reminiscent of the "revenue equivalence theorem," it establishes a relationship that is deeper and more directly based on strategic considerations. See (Szentes, 2001) for more about this method.

The second novelty is the use of mixtures whose supports are surfaces of tetrahedra and whose best-response sets are the tetrahedra themselves. The discovery of this was quite fortuitous; that the set of best responses is of higher dimension than the support of the mixture makes finding the mixture through numerical means unlikely if algorithms are based on better- or best-response dynamics. (If such an algorithm started near the equilibrium, it would tend to increase probability on the best response set and therefore

move away from the equilibrium.) Indeed, it was by no means clear to us ex ante what the dimension of the support of the equilibrium mixture would turn out to be.

It is tempting to think in more general terms of symmetric equilibria having convex best-response sets whose boundaries are the supports of the mixtures. But even though the idea of the support surrounding the set of best responses does seem to work, convexity of the best-response set does not in general. To see that the transformed tetrahedra in Sections 3 and 4 are not generally convex sets, take the example of $\beta = 0.2$ (i.e., $\alpha = 1/3$). Two of the extreme points of $T_{0.2}$ are $(0.2, 1, 1)$ and $(1, 0.2, 1)$; and so $(0.6, 0.6, 1)$ is the midpoint of the extremal edge between them. Now from Proposition 4, $p_{0.2}(1) = 1$, $p_{0.2}(0.2) = 1/3$, and $p_{0.2}(0.6) \approx 0.62$. So $(\mathbf{p_{0.2}}(0.2, 1, 1) + \mathbf{p_{0.2}}(1, 0.2, 1))/2 = (2/3, 2/3, 1)$ cannot be the $\mathbf{p_{0.2}}$-image of any point in $T_{0.2}$.

We are also interested in extending our constructions beyond the 2-bidder, 3-object setting of this paper. Generalizing the pure chopstick example to the case of n (> 3) objects with bidders desiring exactly k $(> n/2)$ of them, a simple symmetric equilibrium with a one-dimensional support exists when the number of bidders is sufficiently large; see (Szentes and Rosenthal, 2001). It is also possible to find symmetric equilibria for multi-object cases with two bidders when the marginal valuations are monotone increasing. The equilibria that we can construct in these cases take on simple forms; again the theme of the support surrounding the set of best responses recurs. All of this is the subject of ongoing research.

Of course, whether knowing any of this equilibrium structure is of use to bidders in real auctions is somewhat dubious. When the set of best responses to the equilibrium mixture is so large, what discipline should a bidder put on his bidding strategy? On the other hand, it ought to be that knowing what equilibrium structures look like in theory should inform the bidder at least on where to focus his thoughts.

Finally, we should point out a related class of games that is also of historical interest. "Colonel Blotto" games are two-person zero-sum games of force (or budget) allocation. In the case that seems closest to our pure chopstick game, the two players each have a unit of force to be allocated among three battles. The winner of each battle is the player who assigns more force to it than the opponent does, and the winner of the game is the one who wins two of the three battles. As far as we know, the first symmetric equilibria of this game were constructed by Borel (see (Borel and Ville, 1938); also (Laslier and Picard, 2000)). These both involve two-dimensional supports; and one of them involves first putting a uniform measure on the hemisphere which sits above the disc that is inscribed in the unit simplex of R^3, and then projecting that measure onto that inscribed disc. This generates a measure that is no longer uniform but that has uniform marginals. So there are aspects of our pure chopstick construction that are reminiscent of Borel's disc solution, but the support of the mixture surrounding the set of best responses apparently has no counterpart there.

Acknowledgments

Appendix A

Proof of Proposition 3. We first introduce the following additional notation: $\mathbf{x} = (x_1, x_2, x_3) \in R^3$; $A(\mathbf{x}) = \{\mathbf{x}': \mathbf{p}(\mathbf{x}') = \mathbf{p}(\mathbf{x})\}$; and for any cumulative distribution function G, μ_G is the associated probability measure.

Let $N(\mathbf{x}, \mathbf{y})$ be the expected revenue (profits plus payments) in $\Gamma(\alpha, p)$ if the bid-triple of the player is $\mathbf{x}$ and the bid-triple of the opponent is $\mathbf{y}$. Let $M_i(\mathbf{x}, \mathbf{y})$ be the probability that bid-triple $\mathbf{x}$ wins the ith object against bid-triple $\mathbf{y}$ in $\Gamma(\alpha, p)$. (So if $x_i \neq y_i$, then $M_i(\mathbf{x}, \mathbf{y})$ is one if x_i is bigger than y_i and zero otherwise.) Then the expected payoff of a player bidding $\mathbf{x}$ against the opponent bidding $\mathbf{y}$ in $\Gamma(\alpha, p)$ is

$$W_p(\mathbf{x}, \mathbf{y}) \equiv N(\mathbf{x}, \mathbf{y}) - \sum_{I=1}^{3} M_i(\mathbf{x}, \mathbf{y}) p(x_i).$$

Let the expected payoff to a player bidding the triple $\mathbf{x}$ in $\Gamma(\alpha, p)$ against μ_{G_p} be

$$U_p(\mathbf{x}) \equiv \int_{R^3} W_p(\mathbf{x}, \mathbf{y}) \, d\mu_{G_p}.$$

The key to the proof is simply noticing that in $\Gamma(\alpha, p)$ randomizing on $\operatorname{supp} \mu_{G_p}$ according to μ_{G_p} is equivalent to randomizing first on the image under $\mathbf{p}$ of $\operatorname{supp} \mu_{G_p}$ according to μ_{G_I} and then for each $\mathbf{a}$ in $\operatorname{supp} \mu_{G_I}$ randomizing on $\{\mathbf{x}: \mathbf{p}(\mathbf{x}) = \mathbf{a}\}$ according to the restriction of μ_{G_p} to this set, that is according to the conditional probability measure

$$\mu_{\mathbf{a}} \equiv \frac{\mu_{G_p}}{\mu_{G_p}(\{\mathbf{x}: \mathbf{p}(\mathbf{x}) = \mathbf{a}\})}.$$

(The measure $\mu_{\mathbf{a}}$ is not yet defined when the denominator above is zero.)

There are two possibilities:

(i) $\mu_{\mathbf{a}}$ can be any measure on the intersection if

$$\operatorname{supp} \mu_{G_p} \cap \{\mathbf{x}: \mathbf{p}(\mathbf{x}) = \mathbf{a}\} \neq \emptyset;$$

(ii) $\mu_{\mathbf{a}}$ can be any measure on $\{\mathbf{x}: \mathbf{p}(\mathbf{x}) = \mathbf{a}\}$ if

$$\operatorname{supp} \mu_{G_p} \cap \{\mathbf{x}: \mathbf{p}(\mathbf{x}) = \mathbf{a}\} = \emptyset.)$$

Therefore $d\mu_{G_p} = d\mu_{\mathbf{a}} \, d\mu_{G_I}$, and we rewrite

$$U_p(\mathbf{x}) = \int_{R^3} \int_{\{\mathbf{y}: \, \mathbf{p}(\mathbf{y}) = \mathbf{a}\}} W_p(\mathbf{x}, \mathbf{y}) \, d\mu_{\mathbf{a}}(\mathbf{y}) \, d\mu_{G_I}(\mathbf{a}).$$

If G_p generates a symmetric equilibrium for $\Gamma(\alpha, p)$, then this expression must be constant on $\operatorname{supp} \mu_{G_p}$ and weakly smaller outside this support.

Now we are ready to define the needed class of tie-break rules in $\Gamma(\alpha, I)$. The idea is: For bid-triples whose inverse images under $\mathbf{p}$ are not single-valued, the tie-break probabilities in $\Gamma(\alpha, I)$ are inherited according to the equilibrium probabilities in $\Gamma(\alpha, p)$. Formally, if the ith coordinates of $\mathbf{p}(\mathbf{x})$ and $\mathbf{p}(\mathbf{y})$ are the same, then the probability that x_i wins is

$$\overline{M}_i\big(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y})\big) \equiv \int_{\mathbf{y}' \in A(\mathbf{y})} \int_{\mathbf{x}' \in A(\mathbf{x})} M_i(\mathbf{x}', \mathbf{y}') \, d\mu_{\mathbf{p}(\mathbf{x})} \, d\mu_{\mathbf{p}(\mathbf{y})}.$$

(For ties outside the range of $\mathbf{p}$, no tie-break rule need be specified.) Expected revenue is therefore

$$\overline{N}\big(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y})\big) = \int\limits_{\mathbf{y}' \in A(\mathbf{y})} \int\limits_{\mathbf{x}' \in A(\mathbf{x})} N(\mathbf{x}', \mathbf{y}') \, d\mu_{\mathbf{p}(\mathbf{x})} \, d\mu_{\mathbf{p}(\mathbf{y})}.$$

So the expected payoff to a bidder bidding $\mathbf{p}(\mathbf{x})$ against $\mathbf{p}(\mathbf{y})$ in $\Gamma(\alpha, I)$ is:

$$W_I\big(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y})\big) = \int\limits_{\mathbf{y}' \in A(\mathbf{y})} \int\limits_{\mathbf{x}' \in A(\mathbf{x})} \left(N(\mathbf{x}', \mathbf{y}') - \sum_{i=1}^{3} M_i(\mathbf{x}', \mathbf{y}') p\big(x_i'\big) \right) d\mu_{\mathbf{p}(\mathbf{x})} \, d\mu_{\mathbf{p}(\mathbf{y})},$$

and the expected payoff from bidding $\mathbf{p}(\mathbf{x})$ is:

$$U_I\big(\mathbf{p}(\mathbf{x})\big) = \int\limits_{R^3} W_I\big(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y})\big) \, d\mu_{G_I}.$$

Applying Fubini's Theorem we get:

$$U_I\big(\mathbf{p}(\mathbf{x})\big) = \int\limits_{\mathbf{x}' \in A(\mathbf{x})} \int\limits_{R^3} \int\limits_{\mathbf{y}' \in A(\mathbf{y})} \left(N(\mathbf{x}', \mathbf{y}') - \sum_{i=1}^{3} M_i(\mathbf{x}', \mathbf{y}') p\big(x_i'\big) \right) d\mu_{\mathbf{p}(\mathbf{y})} \, d\mu_{G_I} \, d\mu_{\mathbf{p}(\mathbf{x})}.$$

Observe that the outermost integrand is just $U_p(\mathbf{x}')$, so $U_I(\mathbf{p}(\mathbf{x}))$ can be rewritten as

$$U_I\big(\mathbf{p}(\mathbf{x})\big) = \int\limits_{\mathbf{x}' \in A(\mathbf{x})} U_p(\mathbf{x}') \, d\mu_{\mathbf{p}(\mathbf{x})}. \tag{A.1}$$

But U_p is constant on the support of μ_{G_p}, and therefore so is U_I on the image under $\mathbf{p}$ of that support, which is the support of μ_{G_I}. It is also clear that if $\mathbf{p}(\mathbf{x}') \in \text{range}(\mathbf{p}) \setminus \text{supp}\, \mu_{G_I}$, $U_I(\mathbf{p}(\mathbf{x}'))$ can be no larger than the same constant, since otherwise $\mathbf{x}'$ would have been a profitable deviation in $\Gamma(\alpha, p)$.

There remains the possibility of a deviation to a triple (z_1, z_2, z_3) that includes a bid, say z_1, outside (i.e., below) the range of p. Since such a bid wins with probability zero, it only remains to show that its companions do not earn positive profits. To have a chance, both of these bids must be in the range of p. But using the tie-break probabilities that are in effect for $(0, p^{-1}(z_2), p^{-1}(z_3))$ in $\Gamma(\alpha, p)$, we see that the deviation cannot be profitable in $\Gamma(\alpha, I)$.　□

Proof of Lemma 5. Assume $(a, b, c) \in T_\beta$ and, without loss of generality, that $a \geqslant b \geqslant c$. For simplicity, we suppress the μ-subscript.

Calculation of $J(a, b, \infty)$: By definition, both bids in the pair (a, b) win against $F3 \cup Ft$ with probability $H(a, b)$. They win against $F2$ with probability $Z(b)$ and against $F1$ with probability $Z(a) - Z(b, a - b)$. The analogous expressions for $J(a, \infty, c)$ and $J(\infty, b, c)$ are derived similarly.

Calculation of $J(a, b, c)$: First, since $(a, b, c) \in T_\beta$, all three bids win all three objects against Ft with probability zero. They win against $F3$ with probability $Z(c)$, against $F2$ with probability $Z(b) - Z(c, b - c)$, and against $F1$ with probability $Z(a) - Z(b, a - b) - Z(c, a - c)$.

Direct substitution now establishes the result.　□

Proof of Lemma 6. Again we suppress the subscripts for simplicity. Let $M = H(a, a) + H(b, b) - 2H(a, b)$. Then $Q(a, b)$ can be rewritten as

$$H(a, a) + H(b, b) + \frac{\alpha}{2}\big[Z(a) + Z(b)\big] + (2 - \alpha)Z(b, a - b) - M.$$

Now M is the $P\mu$-measure of the square generated by the points (a, a) and (b, b) and so is just a constant times $(a - b)^2$. $Z(b, a - b)$ is the measure of the union of two right triangles, both having height $(a - b)$; one having length $(a - b)$, and the other having length $\beta(a - b)$. So $Z(b, a - b)$ is the same constant times $(1 + \beta)(a - b)^2/2$. Since $(1 + \beta) = 2/(2 - \alpha)$, we have

$$Q(a, b) = H(a, a) + H(b, b) + \frac{\alpha}{2}\big[Z(a) + Z(b)\big].$$

Therefore

$$Q(a, b) + Q(a, c) + Q(b, c) = V(a) + V(b) + V(c), \quad \text{where } V(a) = 2H(a, a) + \alpha Z(a). \qquad \Box$$

References

Borel, E., Ville, J., 1938. Application de la théorie des probabilités aux jeux de hasard. Gauthier–Villars, Paris, original edition. Reprinted in: Borel, E., Chéron, A. (Eds.), Théorie mathématique du bridge à la portée de tous. Jacques Gabay, Paris, 1991.

Dasgupta, P., Maskin, E., 1986. The existence of equilibria in discontinuous economic games, I: Theory. Rev. Econ. Stud. 53, 1–27.

Jackson, M., Simon, L., Swinkels, J., Zame, W., 2001. Communication and equilibrium in discontinuous games of incomplete information. Mimeo.

Krishna, V., Rosenthal, R., 1996. Simultaneous auctions with synergies. Games Econ. Behav. 17, 1–31.

Laslier, J.-F., Picard, N., 2000. Distributive politics: Does electoral competition promote inequality? Discussion paper 2000/22, CORE, Universite Catholique de Louvain.

Milgrom, P., 2000. Putting auction theory to work: Ascending auctions with package bidding. Mimeo.

Reny, P., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67, 1029–1056.

Rosenthal, R.W., Wang, R., 1996. Simultaneous auctions with synergies and common values. Games Econ. Behav. 17, 32–55.

Simon, L., Zame, W., 1990. Discontinuous games and endogenous sharing rules. Econometrica 58, 861–872.

Szentes, B., 2001. Equilibrium transformations and the revenue equivalence theorem. Mimeo.

Szentes, B., Rosenthal, R., 2001. Beyond chopsticks: Symmetric equilibria in majority auction games. Mimeo.